

Homework 1

Reading: Treil, Chapter 1, §§1–2.

Note. Do not use any conclusion outside Chapter 1, §§1–2, such as row reduction, determinants, rank, or dimension theorems.

Due date is Sep 24. But note that two homeworks are due on the same day. So you may want to finish this homework earlier.

[LADW] refers to the textbook *Linear Algebra Done Wrong* by Sergei Treil.

1. [LADW] Exercises 1.2(a–d).
2. [LADW] Exercises 1.7.
3. [LADW] Exercises 1.8.
4. [LADW] Exercises 2.1.
5. [LADW] Exercises 2.2.(a–e)
6. [LADW] Exercises 2.3.
7. [LADW] Exercises 2.5.
8. For $s \in \mathbb{R}$, define

$$\mathbf{v}_1 = (1, 0, 1)^T, \quad \mathbf{v}_2 = (0, 1, 1)^T, \quad \mathbf{v}_3 = (1, 1, s)^T.$$

- (a) Determine all s for which the ordered system $\mathcal{B}_s = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis of $\mathbb{R}^3$. For every such s , find formulas for the coefficients in

$$(x, y, z)^T = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3.$$

Explain why the formulas prove both existence and uniqueness.

- (b) Set $s = 3$. Find the coordinate vector of $\mathbf{w} = (2, -1, 4)^T$ with respect to $\mathcal{B}_3$. Verify your answer by substituting into the linear combination.
- (c) For every excluded value of s , describe exactly which $(x, y, z)^T$ are generated by the system, and give one vector that is not generated.

9. Let P_2 be the real vector space of polynomials of degree at most 2, including the zero polynomial. Define

$$q_-(t) = \frac{t(t-1)}{2}, \quad q_0(t) = 1 - t^2, \quad q_+(t) = \frac{t(t+1)}{2}.$$

- (a) Evaluate all three polynomials at -1 , 0 , and 1 . Use those values to prove that q_-, q_0, q_+ are linearly independent.
- (b) Prove that every $p \in P_2$ satisfies

$$p(t) = p(-1)q_-(t) + p(0)q_0(t) + p(1)q_+(t).$$

Deduce that $\mathcal{Q} = (q_-, q_0, q_+)$ is a basis of P_2 .

- (c) For $p(t) = 2 - t + 3t^2$, find its coordinate vectors with respect to $\mathcal{Q}$ and to the standard ordered basis $(1, t, t^2)$. Explain why the coordinate vectors need not agree.