Homework 2

Reading: Treil, Chapter 1, §§3–4.

Note. Do not use any conclusion outside Chapter 1, §§1–4, such as row reduction, determinants, rank, or dimension theorems. Justify your answers.

[LADW] refers to the textbook *Linear Algebra Done Wrong* by Sergei Treil.

1. [LADW] Exercise 3.1(a), (b), (d); p. 16 for (a), (b), and p. 17 for (d).

2. [LADW] Exercise 3.2, p. 17.

3. [LADW] Exercise 3.3(a), (c), (d), p. 17.

4. [LADW] Exercise 3.4(a–c), p. 17. For part (c), take positive rotations to be counterclockwise as viewed from the positive z-axis.

5. [LADW] Exercise 3.5, p. 17.

6. [LADW] Exercise 3.6(a–c), p. 17.

7. [LADW] Exercise 3.7, p. 17.

8. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ with

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad T\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad T\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

(a) Find a formula for $T\begin{pmatrix} x \\ y \end{pmatrix}$ and the matrix of T. Explain why T is uniquely determined.

(b) Could T also satisfy $T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$? Justify your answer.

(c) Define

$$F\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^2 - y^2 \\ 0 \end{pmatrix}.$$

Show that F agrees with T at $\mathbf{0}$, $\mathbf{v}_1$, and $\mathbf{v}_2$, but is not linear. Why does this not contradict part (a)?

9. Let P_1 consist of real polynomials $p(t) = a + bt$. Define

$$S(p) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix}, \qquad T(p) = \begin{pmatrix} p(1) \\ p(-1) \end{pmatrix}.$$

Use the input basis $(1, t)$ and the standard output basis.

(a) Show that S and T are linear and find their matrices.

(b) Let $U(p) = 2T(p) - S(p)$. Find $U(a + bt)$ and the matrix of U.

(c) Show that $V(p) = cS(p) + dT(p)$ is linear for all real c, d. Are S and T linearly independent? Justify your answer.

10. We pretend that we know nothing about quantum mechanics, but we can still do some linear algebra with qubits. We can even prove the famous no-cloning theorem, which is one of the first results indicating that quantum computing is not all mighty and can not do everything.

A *qubit* is a quantum bit. A pure qubit state is represented by a complex column $\mathbf{q} = \begin{pmatrix} a \\ b \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$. Let J be a linear transformation from $\mathbb{C}^2$ to $\mathbb{C}^4$ with

$$J\mathbf{q}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad J\mathbf{q}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We also define the notion of cloning. Cloning a qubit (recall that qubit is represented by a two-dimensional vector $\begin{pmatrix} a \\ b \end{pmatrix}$) means producing the following four-dimensional vector from the two-dimensional vector:

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a^2 \\ ab \\ ab \\ b^2 \end{pmatrix}. \tag{1}$$

Why is this the right definition of cloning? That we will not answer (which is a physics question). But let us just accept this definition and see what we can do with it.

(a) Find the matrix of J and a formula for $J\begin{pmatrix} a \\ b \end{pmatrix}$. Show that J sends normalized vectors to normalized vectors. (Normalized means that the sum of the squares of the absolute values of the entries is 1.)

(b) Let $\mathbf{q}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Compute $J\mathbf{q}_+$, and also compute the vector that would be produced by cloning $\mathbf{q}_+$ according to Eq. (1). Are they equal?

(c) Prove that no linear transformation from $\mathbb{C}^2$ to $\mathbb{C}^4$ can maps every normalized vector $\begin{pmatrix} a \\ b \end{pmatrix}$ to the vector $\begin{pmatrix} a^2 \\ ab \\ ab \\ b^2 \end{pmatrix}$.

(Since quantum gates could only conduct linear operations, this proves that no quantum gate can clone a qubit.)