Homework 3

Reading: Treil, Chapter 1, §§5–6.

Note. Do not use any conclusion outside Chapter 1, §§1–6, such as row reduction, determinants, rank, or dimension theorems. In particular, Remark 6.2 ("an invertible matrix must be square, and a one-sided inverse of a square matrix is automatically two-sided") has not been proved yet and may not be used. To show that a matrix or transformation A is invertible, either exhibit B and check *both* $AB = I$ and $BA = I$, or cite a result from §6 (Theorems 6.3, 6.5, 6.7, 6.8, Corollary 6.9). You may guess an inverse by any method you like, but the guess only counts once both products have been verified.

Due date is Oct 1.

[LADW] refers to the textbook *Linear Algebra Done Wrong* by Sergei Treil.

1. [LADW] Exercises 5.1.
2. [LADW] Exercises 5.3.
3. [LADW] Exercises 5.4.
4. [LADW] Exercises 5.5 and 5.7.
5. [LADW] Exercises 5.6.
6. [LADW] Exercises 6.1.
7. [LADW] Exercises 6.3 and 6.4.
8. [LADW] Exercises 6.5.
9. [LADW] Exercises 6.6 and 6.7.
10. [LADW] Exercises 6.10.
11. Prove or disprove. All matrices are $n \times n$; a disproof requires an explicit counterexample.
    (a) If $AB = 0$ for some matrix $B \neq 0$, then A is not invertible.
    (b) If $A^2 = 0$, then $I - A$ is invertible.
    (c) If A and B are invertible, then $A + B$ is invertible.
    (d) If neither A nor B is invertible, then $A + B$ is not invertible.
    (e) $(A + B)^2 = A^2 + 2AB + B^2$ for all A, B. (If false, state and prove the exact condition on A, B under which the identity holds.)
    (f) If A is invertible and $AB = BA$, then $A^{-1}B = BA^{-1}$.

(g) If A is invertible and $A^T = A$, then A^{-1} is also symmetric.

(h) There exist A, B with $AB - BA = I$.

(i) If $T\colon V \to W$ is an isomorphism and $v_1, \ldots, v_k \in V$ are linearly dependent, then $Tv_1, \ldots, Tv_k$ are linearly dependent.

12. Let $D, S\colon \mathbb{P}_2 \to \mathbb{P}_2$ be the transformations

$$(Dp)(t) = p'(t), \qquad (Sp)(t) = p(t+1).$$

You may take the linearity of D for granted; check the linearity of S in one line. Throughout, use the standard basis $1, t, t^2$ in both the domain and the target.

(a) Find the matrices $[D]$ and $[S]$.

(b) Compute $[D][S]$ and $[S][D]$. What does the result say about the transformations DS and SD? Confirm your conclusion directly from the definitions, without matrices, by computing $(DSp)(t)$ and $(SDp)(t)$ for an arbitrary $p \in \mathbb{P}_2$.

(c) Describe the transformation S^{-1} in words, write down its matrix, and verify by matrix multiplication that $[S][S^{-1}] = [S^{-1}][S] = I$. (Compare the hint to [LADW] Exercise 6.10.)

(d) Compute $[D]^2$ and $[D]^3$. Explain, without computing, why $[D]^3$ had to come out this way. Deduce that D is not invertible, citing the result you use.

13. (Homework 1, Problem 8 revisited.) As in Homework 1, let $v_1 = (1,0,1)^T$, $v_2 = (0,1,1)^T$, $v_3 = (1,1,s)^T$, and let $A_s = [v_1, v_2, v_3]$ be the 3×3 matrix with these columns.

(a) Using Corollary 6.9 and Homework 1, Problem 8(a), determine all $s \in \mathbb{R}$ for which A_s is invertible. No new computation is needed.

(b) For each such s, write down A_s^{-1} explicitly as a 3×3 matrix whose entries depend on s. Hint: by Theorem 6.8, $A_s^{-1}(x,y,z)^T$ is the unique solution $(\alpha,\beta,\gamma)^T$ of $A_s(\alpha,\beta,\gamma)^T = (x,y,z)^T$, and you found formulas for α, β, γ in Homework 1. Verify by multiplication that $A_s A_s^{-1} = I$, and explain why $A_s^{-1} A_s = I$ is then guaranteed without further computation.

(c) Check that $A_3^{-1}(2,-1,4)^T$ agrees with your answer to Homework 1, Problem 8(b). In one sentence: how is the inverse of a matrix related to coordinates with respect to the basis formed by its columns?

(d) For the excluded value of s, show that A_s is neither left invertible nor right invertible. Hint: from Homework 1, Problem 8 you know a nontrivial linear combination of v_1, v_2, v_3 equal to 0, and a vector b that is not generated by v_1, v_2, v_3.

14. (Homework 1, Problem 9 revisited.) Define $T\colon \mathbb{P}_2 \to \mathbb{R}^3$ by

$$T(p) = \big(p(-1),\, p(0),\, p(1)\big)^T.$$

(a) Show that T is linear.

(b) Compute $T(q_-)$, $T(q_0)$, $T(q_+)$ for the polynomials q_-, q_0, q_+ of Homework 1, Problem 9, and use Theorem 6.7 to conclude that T is an isomorphism; hence $\mathbb{P}_2 \cong \mathbb{R}^3$.

(c) Give an explicit formula for $T^{-1}(a,b,c)^T$ as a polynomial, and explain why it is the same statement as Homework 1, Problem 9(b). In words: what does the invertibility of T say about finding a polynomial of degree at most 2 with prescribed values at $t = -1, 0, 1$?

(d) Find the matrix $[T]$ with respect to the bases $1, t, t^2$ of $\mathbb{P}_2$ and e_1, e_2, e_3 of $\mathbb{R}^3$, and the matrix $[T^{-1}]$ with respect to e_1, e_2, e_3 and $1, t, t^2$. Verify by multiplication that $[T][T^{-1}] = I_3$.